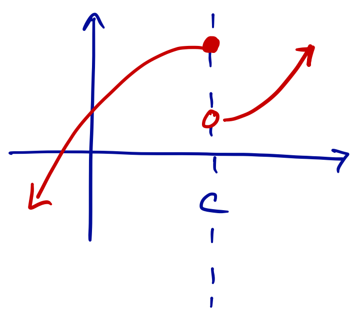
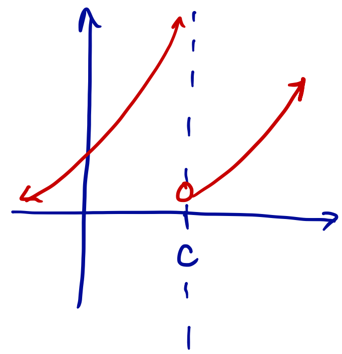
PRACTICE PROBLEMS:

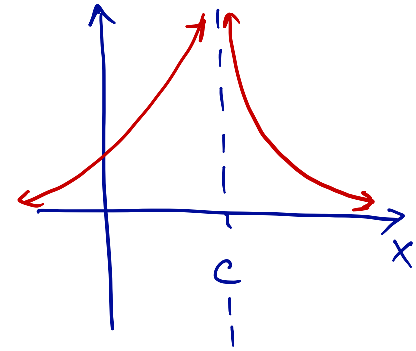
1. Describe several ways in which a limit can fail to exists. Illustrate with sketches.

left + right limits are different





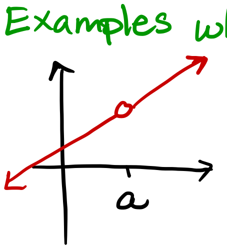
the limit isn't finite



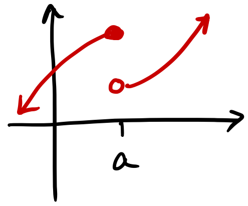
2. Describe what it means for a function $f(x)$ to be continuous at $x = a$ and several ways in which a function $f(x)$ can fail to be continuous at $x = a$. Illustrate with sketches.

The function $f(x)$ is continuous at $x=a$ if it's all one piece at and around $x=a$. That is, the limit approaching a exists and is equal to $f(a)$.

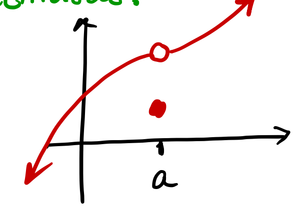
Examples where $f(x)$ fails to be continuous:



removable



no limit

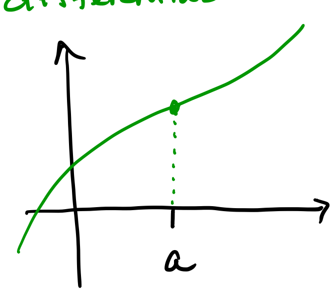


limit not equal to $f(a)$

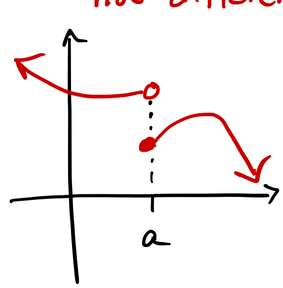
3. Describe what it means for a function $f(x)$ to be differentiable at $x = a$. Illustrate with sketches differentiable and non-differentiable examples.

The function $f(x)$ is differentiable at $x=a$ if $f(x)$ is all one piece at and around $x=a$ (aka f is continuous at a) and f is smooth at $x=a$ (aka no corners)

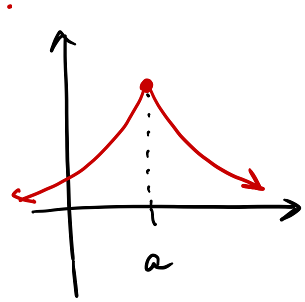
differentiable



not differentiable.

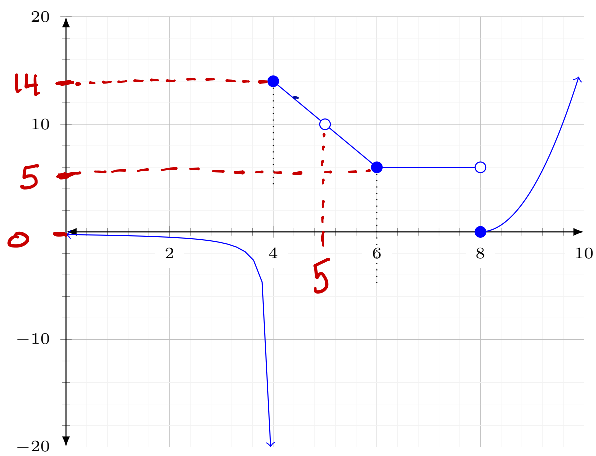


not continuous



continuous, not differentiable.

4. Use the graph of $f(x)$ below to answer the following questions.



(a) Assuming the arrows on the graph indicate a continued curve in that direction, make an educated guess at the domain of the function $f(x)$.

$(-\infty, 5) \cup (5, \infty)$

(b) Find all x-values in the domain of $f(x)$ for which $f(x)$

i. fails to be continuous.

$x = 4, 5, 8$

ii. fails to be differentiable. $x = 4, 5, 6, 8$

(c) Evaluate the following limits or explain why they do not exist.

(i) $\lim_{x\to 4^-} f(x) = -\infty$

(ii) $\lim_{x\to 4^+} f(x) = 14$

(iii) $\lim_{x\to 4} f(x) =$ DNE left + right limits are different

(iv) $\lim_{x\to 5} f(x) = 10$

(v) $\lim_{x\to 6} f(x) = 5$

(vi) $\lim_{x\to 7} f(x) = 5$

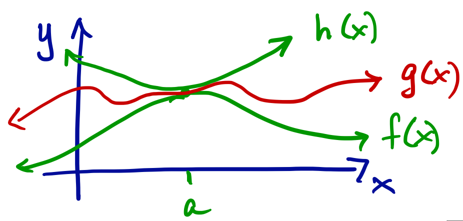
(vi) $\lim_{x\to 8} f(x) =$ DNE. left + right limits are different

(vii) $\lim_{x\to 8^+} f(x) = 0$

5. (a) What does the Squeeze Theorem say? You may want to include a picture with your explanation.

If $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ and $f(x) \le g(x) \le h(x)$, then $\lim_{x\to a} g(x) = L$

Picture:



(b) Use the Squeeze Theorem to show $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$.

thinking:

as $x \to 0^+$, $\frac{\pi}{x} \to +\infty$.

So $\sin(\pi/x)$ oscillates between -1 and $+1$.

So $e^{\sin(\pi/x)}$ oscillates between e^{-1} and $e^{1} = e$.

answer: Choose $f(x) = e^{-1} \cdot \sqrt{x}$ and $h(x) = e\sqrt{x}$.

Observe that

① $\lim_{x\to 0^+} e^{-1}\sqrt{x} = e^{-1} \lim_{x\to 0^+} \sqrt{x} = 0 = \lim_{x\to 0^+} e \cdot \sqrt{x}$.

and ② $e^{-1} \cdot \sqrt{x} \le \sqrt{x} \cdot e^{\sin(\pi/x)} \le e \cdot \sqrt{x}$.

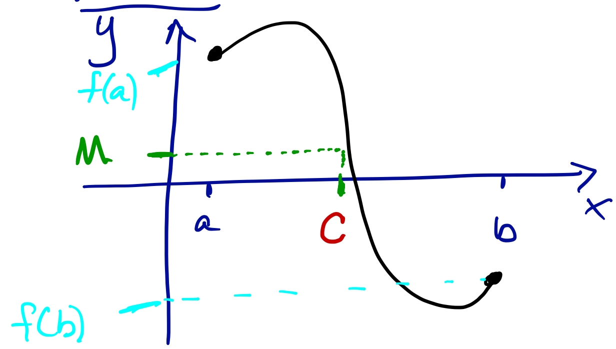
Thus, the Squeeze Theorem implies $\lim_{x\to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0$

6. (a) What does the Intermediate Value Theorem say? You may want to include a picture with your explanation.

If $f(x)$ is continuous on $[a,b]$ and for any M between $f(a)$ and $f(b)$, there is some x-value c in (a,b) where $f(c)=M$.

Alternatively and intuitively, if $f(x)$ is continuous, $f(x)$ must hit every value between $f(a)$ and $f(b)$. No skipping or jumping past y-values.

picture



(b) Use the Intermediate Value Theorem to show $\ln x = x - 5$ has a solution. (Hint: Show there is a solution in the interval $[1, e^5]$.)

Let $f(x) = x - 5 - \ln x$.

Now $f(1) = 1 - 5 - \ln 1 = -4 < 0$

and

$$f(e^5) = e^5 - 5 - \ln(e^5)$$

$$= e^5 - 5 - 5 = e^5 - 10 > 0$$

$e^5 \gg 10$

Also, $f(x)$ is continuous on $(0,\infty)$.

Since 0 is between -4 and $e^5 - 10$, the Intermediate Value Theorem implies that there is an x-value c so that $f(c) = 0$.

So $\ln x = x - 5$ has a solution.

7. (a) Given a function $f(x)$, how do you determine whether or not its graph has any horizontal asymptotes? Give an example of a function $f(x)$ and x-value, c, such that the denominator of $f(x)$ is zero when $x = c$ but $f(x)$ has no vertical asymptote at $x = c$.

Horizontal asymptotes are found by checking what happens for large and small x-values. Specifically, if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ then $y = L$ is a horizontal asymptote.

(b) Given a function $f(x)$, how do you determine whether or not its graph has any vertical asymptotes?

Vertical asymptotes are found by identifying an x-value c so that $\lim_{x\to c^{+-}} f(x) = \pm\infty$

where the limit may approach on either side and the limit itself may approach + or − infinity.

Example: $f(x) = \frac{x-1}{x-1}$ $x = 1$

(c) Find the horizontal and vertical asymptotes (if any) of the graph of $f(x) = \frac{2x^2}{3x^2+2x-1}$.

horizontal:

$$\lim_{x\to\pm\infty} \frac{2x^2}{3x^2+2x-1} = \frac{2}{3}.$$

So $y = \frac{2}{3}$ is a horizontal asymptote.

vertical: $3x^2 + 2x - 1 = (3x-1)(x+1) = 0$.

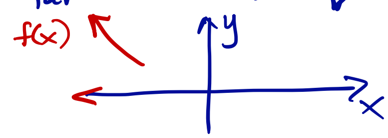
So $x = 1/3$ or $x = -1$.

Now $\lim_{x\to\frac{1}{3}^+} f(x) = \infty$ and $\lim_{x\to-1^+} f(x) = -\infty$

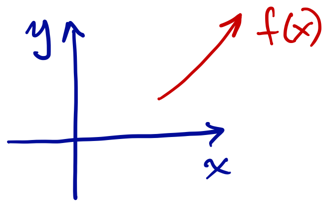
So $x = 1/3$ and $x = -1$ are vertical asymptotes

8. Find the limit or show that it does not exist. In each case, write in your own words, what (if anything) your answers indicate about the graph of the given function.

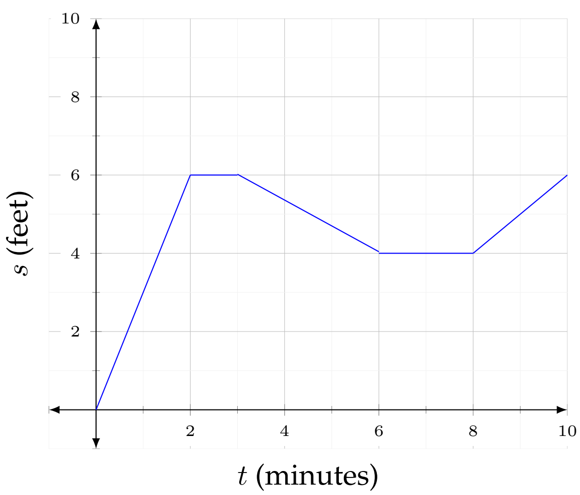
(a) $\lim_{x\to-\infty} \frac{2-x}{3x^2-x} = +\infty$. I know the end behavior of $f(x) = \frac{2-x}{3x^2-x}$ on the "far left". This is



(b) $\lim_{x\to\infty} [\ln(1+x^2) - \ln(1+x)] = \lim_{x\to\infty} \ln\left(\frac{1+x^2}{1+x}\right) = \ln\left(\underbrace{\lim_{x\to\infty} \frac{1+x^2}{1+x}}_{\infty}\right) = \infty$



9. A particle starts by moving to the right along a horizontal line; the graph of its position function is shown in the figure on the below.



(a) At what times is the particle moving to the right?

$0 \le t \le 2$ and $8 \le t \le 10$

(b) At what times is the particle moving to the left?

$3 \le t \le 6$

(c) At what times is the particle standing still?

$2 \le t \le 3$, $6 \le t \le 8$

(d) Sketch a graph of the velocity of the particle.

thinking

- velocity = derivative = slope
- So velocity for each "chunk" is constant
- At "corners" velocity is undefined

interval	[0,2]	[2,3]	[3,6]	[6,8]	[8,10]
slope	3	0	$-\frac{2}{3}$	0	1